

A Lie algebra related to the universal Askey-Wilson algebra

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Abstract

Let \mathbb{F} denote an algebraically closed field. Denote the three-element set by $\mathcal{X} = \{A, B, C\}$, and let $\mathbb{F}\langle\mathcal{X}\rangle$ denote the free unital associative \mathbb{F} -algebra on \mathcal{X} . Fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. The universal Askey-Wilson algebra Δ is the quotient space $\mathbb{F}\langle\mathcal{X}\rangle/\mathbb{I}$, where \mathbb{I} is the two-sided ideal of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by the nine elements $UV - VU$, where U is one of A, B, C , and V is one of

$$\begin{aligned} (q + q^{-1})A + \frac{qBC - q^{-1}CB}{q - q^{-1}}, \\ (q + q^{-1})B + \frac{qCA - q^{-1}AC}{q - q^{-1}}, \\ (q + q^{-1})C + \frac{qAB - q^{-1}BA}{q - q^{-1}}. \end{aligned}$$

Turn $\mathbb{F}\langle\mathcal{X}\rangle$ into a Lie algebra with Lie bracket $[X, Y] = XY - YX$ for all $X, Y \in \mathbb{F}\langle\mathcal{X}\rangle$. Let \mathcal{L} denote the Lie subalgebra of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by \mathcal{X} , which is also the free Lie algebra on \mathcal{X} . Let L denote the Lie subalgebra of Δ generated by A, B, C . Since the given set of defining relations of Δ are not in \mathcal{L} , it is natural to conjecture that L is freely generated by A, B, C . We give an answer in the negative by showing that the kernel of the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \rightarrow \Delta$ has a nonzero intersection with \mathcal{L} . Denote the span of all Hall basis elements of \mathcal{L} of length n by \mathcal{L}_n , and denote the image of $\sum_{i=1}^n \mathcal{L}_i$ under the canonical map $\mathcal{L} \rightarrow L$ by L_n . We show that the simplest nontrivial Lie algebra relations on L occur in L_5 . We exhibit a basis for L_4 , and we also exhibit a basis for L_5 if q is not a sixth root of unity.

Keywords: universal Askey-Wilson algebra; Hall basis; Lie algebra relations

1 Introduction

Let \mathbb{F} be an algebraically closed field and fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. Given $a, b, c \in \mathbb{F}$, the *Askey-Wilson algebra* with parameters a, b, c is the unital associative \mathbb{F} -algebra $AW := AW_q(a, b, c)$ defined as having generators A, B, C and relations

$$\begin{aligned} A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} &= \frac{a}{q + q^{-1}}, \\ B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} &= \frac{b}{q + q^{-1}}, \\ C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} &= \frac{c}{q + q^{-1}}. \end{aligned}$$

The algebra AW was introduced in [8] in order to describe the Askey-Wilson polynomials [2]. A wide range of applications of the Askey-Wilson algebra is discussed in [7, Section 1]. These applications include

integrable systems, quantum mechanics, the theory of quadratic algebras, Leonard pairs and Leonard triples, and quantum groups. A central extension of the Askey-Wilson algebra AW is introduced in [7], which is called the *universal Askey-Wilson algebra*.

Definition 1.1 ([7, Definition 1.2]). *The universal Askey-Wilson algebra is the unital associative \mathbb{F} -algebra, which we denote by Δ , defined as having generators A, B, C , and relations which assert that the following are central in Δ :*

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad (1)$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad (2)$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}, \quad (3)$$

where q is a nonzero scalar that is not a fourth root of unity.

Our main object of study is the Lie subalgebra L of Δ generated by A, B, C . We show that a set of defining relations for Δ cannot be expressed in terms of Lie algebra operations only, and yet this does not imply that L is freely generated by A, B, C . Denote the free unital associative \mathbb{F} -algebra on the three-element set $\mathcal{X} = \{A, B, C\}$ by $\mathbb{F}\langle\mathcal{X}\rangle$, and the free Lie algebra on \mathcal{X} by \mathcal{L} . Recall that \mathcal{L} is the Lie subalgebra of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by A, B, C . We use the basis of \mathcal{L} which was introduced by Hall [5]. Let us call the images of the Hall basis elements under the canonical map $\mathcal{L} \rightarrow L$ as the standard Lie monomials of L . We show that the kernel of the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \rightarrow \Delta$ has a nonzero intersection with \mathcal{L} . The generators A, B, C are the standard Lie monomials of length 1. The standard Lie monomials of lengths ≥ 1 are constructed according to some rules, which we shall discuss in later sections. We show that the simplest Lie algebra relations on L occur at length 5, and we determine a maximal linearly independent set of standard Lie monomials of length at most 5.

2 Preliminaries

Let \mathbb{F} be an algebraically closed field. Throughout, by an \mathbb{F} -algebra we mean a unital associative \mathbb{F} -algebra. Let \mathfrak{A} be an \mathbb{F} -algebra. Recall that an anti-automorphism of \mathfrak{A} is a bijective \mathbb{F} -linear map $\psi : \mathfrak{A} \rightarrow \mathfrak{A}$ such that $\psi(fg) = \psi(g)\psi(f)$ for all $f, g \in \mathfrak{A}$. We turn \mathfrak{A} into a Lie algebra with Lie bracket $[f, g] = fg - gf$ for $f, g \in \mathfrak{A}$.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of natural numbers. Given a nonzero $n \in \mathbb{N}$, let \mathcal{X} denote an n -element set. We shall refer to any element of \mathcal{X} as a *letter*. For $t \in \mathbb{N}$, by a *word of length t* on \mathcal{X} we mean a sequence of the form

$$X_1 X_2 \cdots X_t, \quad (4)$$

where $X_i \in \mathcal{X}$ for $1 \leq i \leq t$. Given a word W on \mathcal{X} , denote the length of W by $|W|$. The word of length 0 will be denoted by 1. Let $\langle\mathcal{X}\rangle$ denote the set of all words on \mathcal{X} . Given words $X_1 X_2 \cdots X_s$ and $Y_1 Y_2 \cdots Y_t$ on \mathcal{X} , their *concatenation product* is

$$X_1 X_2 \cdots X_s Y_1 Y_2 \cdots Y_t.$$

We now recall the free \mathbb{F} -algebra $\mathbb{F}\langle\mathcal{X}\rangle$. The \mathbb{F} -vector space $\mathbb{F}\langle\mathcal{X}\rangle$ has basis $\langle\mathcal{X}\rangle$. Multiplication in the \mathbb{F} -algebra $\mathbb{F}\langle\mathcal{X}\rangle$ is the concatenation product. We endow $\mathbb{F}\langle\mathcal{X}\rangle$ with a symmetric bilinear form $(\ , \)$ with respect to which $\langle\mathcal{X}\rangle$ is an orthonormal basis. For any $f \in \mathbb{F}\langle\mathcal{X}\rangle$ and any word W , the coefficient of W in f is (f, W) .

Given $n \in \mathbb{N}$, the subspace of $\mathbb{F}\langle\mathcal{X}\rangle$ spanned by all the words of length n is the *n -homogenous component* of $\mathbb{F}\langle\mathcal{X}\rangle$. Observe that $\mathbb{F}\langle\mathcal{X}\rangle$ is the direct sum of all the n -homogenous components for $n \in \mathbb{N}$. If f is an element of the m -homogenous component and g is an element of the n -homogenous component, then fg is an element of the $(m+n)$ -homogenous component. It follows that the set of all n -homogenous components of $\mathbb{F}\langle\mathcal{X}\rangle$ for all $n \in \mathbb{N}$ is a grading of $\mathbb{F}\langle\mathcal{X}\rangle$.

The following notation will be useful. Let $W = X_1 X_2 \cdots X_t$ denote a word on \mathcal{X} . We define W^* to be the word $X_t X_{t-1} \cdots X_1$ on \mathcal{X} . Let θ denote the \mathbb{F} -linear map

$$\begin{aligned} \theta : \mathbb{F}\langle \mathcal{X} \rangle &\rightarrow \mathbb{F}\langle \mathcal{X} \rangle, \\ W &\mapsto (-1)^{|W|} W^*, \end{aligned} \tag{5}$$

for any word W . By [6, p. 19], the map θ is the unique anti-automorphism of the \mathbb{F} -algebra $\mathbb{F}\langle \mathcal{X} \rangle$ that sends X to $-X$ for any letter X .

Let \mathcal{L} denote the Lie subalgebra of the Lie algebra $\mathbb{F}\langle \mathcal{X} \rangle$ generated by \mathcal{X} . Following [6, Theorem 0.5], we call \mathcal{L} the *free Lie algebra on \mathcal{X}* .

Proposition 2.1 ([6, Lemma 1.7]). *For $f \in \mathcal{L}$, we have $\theta(f) = -f$.*

We now recall the notion of a *Lie monomial* on \mathcal{X} . The set of all Lie monomials on \mathcal{X} is the minimal subset of $\mathbb{F}\langle \mathcal{X} \rangle$ that contains \mathcal{X} and is closed under the Lie bracket. Observe that 0 is a Lie monomial. Let U be a Lie monomial. Then U is an element of some n -homogenous component of $\mathbb{F}\langle \mathcal{X} \rangle$. We define the *length* of the Lie monomial U to be n . Observe that 0 has length n for any $n \in \mathbb{N}$. Any nonzero Lie monomial has a unique length. Observe that the set of all Lie monomials of length 1 is \mathcal{X} . We now consider an ordering of Lie monomials.

Definition 2.2 ([4, p. 581]). *Fix an ordering $<$ on \mathcal{X} . Suppose that the set of all Lie monomials of lengths $1, 2, \dots, t-1$ have been ordered such that $U < V$ if the length of U is strictly less than that of V . If U, V both have length t , and can be written as $U = [X_1, Y_1], V = [X_2, Y_2]$, then we compare U, V using the following rules:*

- (i) *If $Y_1 \neq Y_2$, then $U < V$ iff $Y_1 < Y_2$.*
- (ii) *If $Y_1 = Y_2$, then $U < V$ iff $X_1 < X_2$.*

We now introduce a basis for \mathcal{L} consisting of Lie monomials.

Proposition 2.3 ([5, Theorem 3.1]). *Let \mathbb{H} be the set of Lie monomials such that $\mathcal{X} \subset \mathbb{H}$, and that for any $U, V \in \mathbb{H}$, the Lie monomial $[U, V]$ is also in \mathbb{H} whenever the following conditions hold.*

- (i) *$U > V$.*
- (ii) *If $U = [X, Y]$ for some Lie monomials X, Y , then $Y \leq V$.*

Then \mathbb{H} is a basis for \mathcal{L} , often referred to as the Hall basis of \mathcal{L} .

Example 2.4. *Suppose $\mathcal{X} = \{A, B, C\}$ and $A < B < C$. Then the elements of \mathbb{H} of length at most 4 are:*

$$\begin{aligned} &A, B, C, [B, A], [C, A], [C, B], [[B, A], A], [[C, A], A], [[B, A], B], \\ &\quad [[C, A], B], [[C, B], B], [[B, A], C], [[C, A], C], [[C, B], C], \\ &\quad [[[B, A], A], A], [[[C, A], A], A], [[[B, A], A], B], [[[C, A], A], B], \\ &\quad [[[B, A], B], B], [[[C, A], B], B], [[[C, B], B], B], [[[B, A], A], C], \\ &\quad [[[C, A], A], C], [[[B, A], B], C], [[[C, A], B], C], [[[C, B], B], C], \\ &\quad [[[B, A], C], C], [[[C, A], C], C], [[[C, B], C], C], \\ &\quad [[C, A], [B, A]], [[C, B], [B, A]], [[C, B], [C, A]]. \end{aligned} \tag{6}$$

Observe that the above Lie monomials are listed according to the ordering in Definition 2.2.

Given a Lie algebra \mathfrak{L} and $x, y \in \mathfrak{L}$, recall the adjoint linear map

$$\text{ad } x : \mathfrak{L} \rightarrow \mathfrak{L}$$

that sends $y \mapsto [x, y]$. Denote an arbitrary word on \mathcal{X} by $W = X_1 X_2 \cdots X_t$. The *Lie bracketing from left to right* is the linear map $\mathbb{F}\langle \mathcal{X} \rangle \rightarrow \mathcal{L}$ that sends $1 \mapsto 0$ and sends the word W into a Lie monomial according to the following rules:

(i) If $|W| = 1$, then $W \mapsto W$.

(ii) Suppose that the images of all words of length $< |W|$ have been defined. Denote the image of $X_1 X_2 \cdots X_{t-1}$ by V . Then

$$W \mapsto (-\text{ad } X_t)(V) = [V, X_t].$$

That is, $X_1 X_2 \cdots X_t \mapsto [[[X_1, X_2], \cdots], X_t]$ for $t \geq 2$. A Lie monomial that is an image of some word under Lie bracketing from left to right is said to be *left-normed*.

Notation 2.5. Given a word W , we denote the image of W under Lie bracketing from left to right by $[W]$.

Example 2.6. With reference to Example 2.4, we rewrite (6) using Notation 2.5.

$$\begin{aligned} A, B, C, [BA], [CA], [CB], [BA^2], [CA^2], [BAB], \\ [CAB], [CB^2], [BAC], [CAC], [CBC], \\ [BA^3], [CA^3], [BA^2B], [CA^2B], \\ [BAB^2], [CAB^2], [CB^3], [BA^2C], \\ [CA^2C], [BABC], [CABC], [CB^2C], \\ [BAC^2], [CAC^2], [CBC^2], \\ [[CA], [BA]], [[CB], [BA]], [[CB], [CA]]. \end{aligned} \quad (7)$$

Throughout, by an *ideal* of an \mathbb{F} -algebra \mathfrak{A} we mean a two-sided ideal of \mathfrak{A} . By a *Lie ideal* of a Lie algebra \mathfrak{L} we mean an ideal of \mathfrak{L} under the Lie algebra structure. We now recall the notion of algebras having generators and relations (i.e., having a presentation). Denote the elements of \mathcal{X} by G_1, G_2, \dots, G_n .

Let $f_1, f_2, \dots, f_m \in \mathbb{F}\langle \mathcal{X} \rangle$ and let I be the ideal of $\mathbb{F}\langle \mathcal{X} \rangle$ generated by f_1, f_2, \dots, f_m . We define $\mathbb{F}\langle \mathcal{X} \rangle / I$ as the \mathbb{F} -algebra with generators G_1, G_2, \dots, G_n and relations $f_1 = 0, f_2 = 0, \dots, f_m = 0$. The Lie subalgebra of $\mathbb{F}\langle \mathcal{X} \rangle / I$ generated by \mathcal{X} is $\mathcal{L} / (I \cap \mathcal{L})$.

Let $g_1, g_2, \dots, g_m \in \mathcal{L}$ and let J be the Lie ideal of \mathcal{L} generated by g_1, g_2, \dots, g_m . We define \mathcal{L} / J as the Lie algebra with generators G_1, G_2, \dots, G_n and relations $g_1 = 0, g_2 = 0, \dots, g_m = 0$.

Suppose \mathfrak{L} is a Lie algebra (over \mathbb{F}) generated by \mathcal{X} . Then there exists an ideal \mathcal{K} of \mathcal{L} such that $\mathfrak{L} = \mathcal{L} / \mathcal{K}$. Let $\phi : \mathcal{L} \rightarrow \mathcal{L} / \mathcal{K}$ be the canonical Lie algebra homomorphism. Then the following span \mathfrak{L} :

$$\phi(U), \quad \text{for } U \in \mathbb{H}. \quad (8)$$

We call (8) the *standard Lie monomials* of the Lie algebra \mathfrak{L} . Observe that the list of the standard Lie monomials of \mathfrak{L} is identical to the list of elements of \mathbb{H} . This is because the Lie algebra homomorphism ϕ fixes generators. We order the list of standard Lie monomials of \mathfrak{L} in a manner analogous to that given in Definition 2.2.

3 The universal Askey-Wilson algebra

Hereon, let \mathbb{F} be an algebraically closed field, and fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. We fix $\mathcal{X} = \{A, B, C\}$. Let $\mathbb{F}\langle \mathcal{X} \rangle$ be the free associative algebra on \mathcal{X} . We use the ordering $A < B < C$ to construct the Hall basis \mathbb{H} of the free Lie algebra \mathcal{L} on \mathcal{X} . Define the following elements of the free algebra $\mathbb{F}\langle \mathcal{X} \rangle$.

$$\alpha := (q + q^{-1})A + \frac{qBC - q^{-1}CB}{q - q^{-1}}, \quad (9)$$

$$\beta := (q + q^{-1})B + \frac{qCA - q^{-1}AC}{q - q^{-1}}, \quad (10)$$

$$\gamma := (q + q^{-1})C + \frac{qAB - q^{-1}BA}{q - q^{-1}}. \quad (11)$$

We also define the following Lie products in $\mathbb{F}\langle\mathcal{X}\rangle$.

$$\begin{aligned} r_0 &:= [A, \alpha], & r_3 &:= [B, \alpha], & r_6 &:= [C, \alpha], \\ r_1 &:= [B, \beta], & r_4 &:= [C, \beta], & r_7 &:= [A, \beta], \\ r_2 &:= [C, \gamma], & r_5 &:= [A, \gamma], & r_8 &:= [B, \gamma]. \end{aligned}$$

Define \mathbb{I} as the ideal of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by r_0, r_1, \dots, r_8 .

With reference to Definition 1.1, we express Δ as a quotient space of $\mathbb{F}\langle\mathcal{X}\rangle$, and as a consequence make explicit the defining relations of Δ .

Proposition 3.1. $\Delta = \mathbb{F}\langle\mathcal{X}\rangle / \mathbb{I}$.

Proof. Recall Δ has relations which assert that each of (1),(2),(3) commutes with every element of Δ . Equivalently, each of (1),(2),(3) commutes with every generator A, B, C . Observe that each of α, β, γ is a scalar multiple of (1),(2),(3), respectively. Then it suffices to define Δ as having nine defining relations of the form $[X, \delta]$, where $X \in \{A, B, C\}$ and $\delta \in \{\alpha, \beta, \gamma\}$. By the definition of \mathbb{I} , we get the desired result. ■

We denote the images of α, β, γ under the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \rightarrow \Delta$ by the same symbols.

Proposition 3.2. $r_0, r_1, \dots, r_8 \notin \mathcal{L}$.

Proof. Let θ denote the \mathbb{F} -linear map in (5). It is routine to show that in the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$, we have $\theta(r_i) + r_i \neq 0$ for $0 \leq i \leq 8$. Use Proposition 2.1. ■

By a *word in Δ* we mean the image of an element of $\langle\mathcal{X}\rangle$ under the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \rightarrow \Delta$. Observe that the list of all words in Δ is identical to the list of all the words on \mathcal{X} in the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$ since the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \rightarrow \Delta$ is an \mathbb{F} -algebra homomorphism that fixes generators. We also preserve the ordering of generators $A < B < C$ in Δ . By a Δ -word, we mean all elements of Δ of the form

$$W\alpha^r\beta^s\gamma^t \tag{12}$$

where W is a word in Δ , and $r, s, t \in \mathbb{N}$.

We now recall some properties of Δ as studied in [7]. Let $U = X_1X_2 \cdots X_t$ be a Δ -word, where X_i is either a generator of Δ or one of α, β, γ for $1 \leq i \leq t$. Without loss of generality, we assume U is of the form (12) since α, β, γ are central in Δ . By an *inversion* for W we mean an ordered pair $(j, k) \in \mathbb{N}^2$ such that $1 \leq j < k \leq t$ and $X_j, X_k \in \{A, B, C\}$ such that $X_j > X_k$. Any Δ -word with no inversions is said to be *irreducible*. For instance, $CABA$ has 4 inversions and CB^2A has 5, while the Δ -words A^2BC, AB^2C are irreducible. The shortest words for which inversions exist are BA, CA, CB and using (9) to (11), the following hold in both $\mathbb{F}\langle\mathcal{X}\rangle$ and Δ .

$$BA = q^2AB + q(q + q^{-1})(q - q^{-1})C - q(q - q^{-1})\gamma, \tag{13}$$

$$CA = q^{-2}AC - q^{-1}(q + q^{-1})(q - q^{-1})B + q^{-1}(q - q^{-1})\beta, \tag{14}$$

$$CB = q^2BC + q(q + q^{-1})(q - q^{-1})A - q(q - q^{-1})\alpha. \tag{15}$$

Consider the word $CABA$, one of the 4 inversions in which is caused by the first two letters C, A . Substituting for CA using (14), the result is a linear combination of $ACBA, B^2A, BA\beta$, each having fewer inversions than $CABA$.

Remark 3.3. Following [7, p. 7] and [3, Theorem 1.2], for any Δ -word W , there exists a finite number of steps of substituting for inversions using (13) to (15) such that the final result is a unique linear combination of irreducible Δ -words. It follows that a basis for Δ consists of the vectors

$$A^iB^jC^k\alpha^r\beta^s\gamma^t, \quad i, j, k, r, s, t, \in \mathbb{N}. \tag{16}$$

Given subspaces H, K of Δ , define $HK := \text{Span} \{hk \mid h \in H, k \in K\}$. If K is a subspace of H , we say that a subspace K' of H is a *complement of K in H* whenever

$$H = K + K'. \quad (\text{direct sum})$$

We now recall a filtration for Δ as given in [7, Section 5]. This filtration is a sequence $\{\Delta_n\}_{n \in \mathbb{N}}$ of subspaces of Δ defined by

$$\begin{aligned}\Delta_0 &:= \mathbb{F}1, \\ \Delta_1 &:= \Delta_0 + \text{Span} \{A, B, C, \alpha, \beta, \gamma\}, \\ \Delta_n &:= \Delta_1 \Delta_{n-1}, \quad n \geq 1,\end{aligned}$$

and has the following properties for all $i, j \in \mathbb{N}$.

$$\Delta_i \subseteq \Delta_{i+1}, \quad (17)$$

$$\Delta = \bigcup_{n \in \mathbb{N}} \Delta_n,$$

$$\Delta_i \Delta_j = \Delta_{i+j}. \quad (18)$$

Given $n \in \mathbb{N}$, a basis for Δ_n consists of the vectors

$$A^i B^j C^k \alpha^r \beta^s \gamma^t, \quad i, j, k, r, s, t \in \mathbb{N}, \quad i + j + k + r + s + t \leq n, \quad (19)$$

while the following vectors form a basis for a complement of Δ_n in Δ_{n+1}

$$A^i B^j C^k \alpha^r \beta^s \gamma^t, \quad i, j, k, r, s, t \in \mathbb{N}, \quad i + j + k + r + s + t = n + 1. \quad (20)$$

We denote the span of the vectors (20) by Δ_n^c .

By [7, Lemma 6.1], the following elements of Δ coincide and are central.

$$qABC + q^2 A^2 + q^{-2} B^2 + q^2 C^2 - qA\alpha - q^{-1} B\beta - qC\gamma, \quad (21)$$

$$qBCA + q^2 A^2 + q^2 B^2 + q^{-2} C^2 - qA\alpha - qB\beta - q^{-1} C\gamma, \quad (22)$$

$$qCAB + q^{-2} A^2 + q^2 B^2 + q^2 C^2 - q^{-1} A\alpha - qB\beta - qC\gamma, \quad (23)$$

$$q^{-1} CBA + q^{-2} A^2 + q^2 B^2 + q^{-2} C^2 - q^{-1} A\alpha - qB\beta - q^{-1} C\gamma, \quad (24)$$

$$q^{-1} ACB + q^{-2} A^2 + q^{-2} B^2 + q^2 C^2 - q^{-1} A\alpha - q^{-1} B\beta - qC\gamma, \quad (25)$$

$$q^{-1} BAC + q^2 A^2 + q^{-2} B^2 + q^{-2} C^2 - qA\alpha - q^{-1} B\beta - q^{-1} C\gamma. \quad (26)$$

Denote this element by Ω , which is called in [7] as the *Casimir element of Δ* . As shown in [7, Section 7], we have other bases for Δ, Δ_n (for $n \in \mathbb{N}$) that involve Ω . First, the following vectors form a basis for Δ .

$$A^i B^j C^k \Omega^l \alpha^r \beta^s \gamma^t, \quad i, j, k, l, r, s, t \in \mathbb{N}, \quad ijk = 0. \quad (27)$$

Given $n \in \mathbb{N}$, a basis for Δ_n consists of the vectors

$$A^i B^j C^k \Omega^l \alpha^r \beta^s \gamma^t, \quad i, j, k, l, r, s, t \in \mathbb{N}, \quad ijk = 0, \quad i + j + k + 3l + r + s + t \leq n, \quad (28)$$

while the following vectors form a basis for a complement of Δ_n in Δ_{n+1} .

$$A^i B^j C^k \Omega^l \alpha^r \beta^s \gamma^t, \quad i, j, k, l, r, s, t \in \mathbb{N}, \quad ijk = 0, \quad i + j + k + 3l + r + s + t = n + 1.$$

Recall that Δ is a Lie algebra with Lie bracket $[X, Y] := XY - YX$ for $X, Y \in \Delta$. Denote the derived algebra of Δ by $[\Delta, \Delta]$, and the ideal of Δ generated by $[\Delta, \Delta]$ by $\Delta[\Delta, \Delta]\Delta$. It follows that the Lie subalgebra of Δ generated by A, B, C is $L := \mathcal{L}/(\mathbb{I} \cap \mathcal{L})$. Given nonzero $n \in \mathbb{N}$, denote the span of all Hall basis elements of \mathcal{L} of length n by \mathcal{L}_n . Denote the image of $\sum_{i=1}^n \mathcal{L}_i$ under the canonical map $\mathcal{L} \rightarrow L$ by L_n . It follows that all standard Lie monomials of L of length at most n span L_n .

Proposition 3.4. $L \subseteq \mathbb{F}A + \mathbb{F}B + \mathbb{F}C + \Delta[\Delta, \Delta]\Delta$.

Proof. By the definition of L , we have $L \subseteq \mathbb{F}A + \mathbb{F}B + \mathbb{F}C + [\Delta, \Delta]$. Since Δ has a multiplicative identity, we have $[\Delta, \Delta] \subseteq \Delta[\Delta, \Delta]\Delta$. From these we get the desired set inclusion. \blacksquare

Proposition 3.5. If q is not a root of unity, then L has zero center.

Proof. Let q be not a root of unity. Suppose that $Z(L)$ has a nonzero element f . Since the generators A, B, C of Δ are also in L , we have $Z(L) \subseteq Z(\Delta)$. By [7, Corollary 8.3], $Z(\Delta)$ is generated by $\alpha, \beta, \gamma, \Omega$. Observe that there exists a filtration subspace Δ_n such that $f \in \Delta_n$, and that $\Delta_n \cap Z(\Delta)$ has a basis consisting of the vectors

$$\Omega^l \alpha^r \beta^s \gamma^t, \quad l, r, s, t \in \mathbb{N}, \quad 3l + r + s + t \leq n. \quad (29)$$

Since f is nonzero, there exists a nonzero $c \in \mathbb{F}$ and a vector $\Omega^w \alpha^x \beta^y \gamma^z$ in (29) such that

$$f - c\Omega^w \alpha^x \beta^y \gamma^z = g, \quad (30)$$

where g is a linear combination of vectors in (29) other than $\Omega^w \alpha^x \beta^y \gamma^z$. Let $\mathbb{F}[\bar{A}, \bar{B}, \bar{C}]$ denote the \mathbb{F} -algebra of polynomials in three mutually commuting indeterminates $\bar{A}, \bar{B}, \bar{C}$, with coefficients from \mathbb{F} . As shown in [7, p. 17], there exists a unique surjective \mathbb{F} -algebra homomorphism $\Psi : \Delta \rightarrow \mathbb{F}[\bar{A}, \bar{B}, \bar{C}]$ with kernel $\Delta[\Delta, \Delta]\Delta$ that sends

$$A \mapsto \bar{A}, \quad B \mapsto \bar{B}, \quad C \mapsto \bar{C}. \quad (31)$$

Under this homomorphism, denote the images of $\alpha, \beta, \gamma, \Omega$ by $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\Omega}$, respectively. As shown in [7, Lemma 11.3], we have

$$\bar{\alpha} = (q + q^{-1})\bar{A} + \bar{B}\bar{C}, \quad (32)$$

$$\bar{\beta} = (q + q^{-1})\bar{B} + \bar{A}\bar{C}, \quad (33)$$

$$\bar{\gamma} = (q + q^{-1})\bar{C} + \bar{A}\bar{B}, \quad (34)$$

$$\bar{\Omega} = (q + q^{-1})\bar{A}\bar{B}\bar{C} - \bar{A}^2 - \bar{B}^2 - \bar{C}^2. \quad (35)$$

It is routine to show that the vectors

$$\bar{A}, \bar{B}, \bar{C}, \bar{\Omega}^l \bar{\alpha}^r \bar{\beta}^s \bar{\gamma}^t, \quad l, r, s, t \in \mathbb{N}, \quad 3l + r + s + t \leq n, \quad (36)$$

are linearly independent in $\mathbb{F}[\bar{A}, \bar{B}, \bar{C}]$. Observe also that by Proposition 3.4,

$$f \in \mathbb{F}A + \mathbb{F}B + \mathbb{F}C + \ker \Psi.$$

Applying Ψ to both sides of (30), we have

$$c_1 \bar{A} + c_2 \bar{B} + c_3 \bar{C} - c\bar{\Omega}^w \bar{\alpha}^x \bar{\beta}^y \bar{\gamma}^z = \bar{g}, \quad (37)$$

where \bar{g} is a linear combination of the vectors in (36) except $\bar{\Omega}^w \bar{\alpha}^x \bar{\beta}^y \bar{\gamma}^z$. We get a contradiction from (37). Therefore, $Z(L) = 0$. \blacksquare

We end this section by discussing some properties of Δ related to the group $PSL_2(\mathbb{Z})$. We denote by $PSL_2(\mathbb{Z})$ the free product of the cyclic group of order two and the cyclic group of order three [1]. Let ρ, σ denote the generators of $PSL_2(\mathbb{Z})$ such that $\rho^3 = 1$ and $\sigma^2 = 1$. By [7, Theorem 3.1], the group $PSL_2(\mathbb{Z})$ acts faithfully on Δ as a group of automorphisms in the following way:

u	A	B	C	α	β	γ
$\rho(u)$	B	C	A	β	γ	α
$\sigma(u)$	B	A	$C + (q - q^{-1})^{-1}[A, B]$	β	α	γ

By [7, Theorem 6.4], Ω is fixed by ρ, σ . It is routine to show that given $n \in \mathbb{N}$, the filtration subspace Δ_n is invariant under ρ .

Proposition 3.6. *The Lie algebra L is $PSL_2(\mathbb{Z})$ -invariant.*

Proof. Let $\tau \in \{\rho, \sigma\}$. It suffices to argue in the following way. Show that the images of the generators A, B, C under τ are in L , and show that if the images of $f, g \in L$ under τ are in L , then so is the image of $[f, g]$ under τ . By the above table, we are done with the first step. For the second step, assume that the images of $f, g \in L$ under τ are in L . Since τ is an \mathbb{F} -algebra automorphism, we have $\tau([f, g]) = \tau(fg - gf) = [\tau(f), \tau(g)] \in L$. \blacksquare

4 L is not free

In this section, all computations are done in the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$. Our goal is to show that $\mathbb{I} \cap \mathcal{L}$ contains a nonzero element.

Proposition 4.1. *In the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$,*

$$\frac{[BA]}{q(q-q^{-1})} - (q+q^{-1})C = AB - \gamma, \quad (38)$$

$$\frac{[CA]}{q^{-1}(q-q^{-1})} - (q+q^{-1})B = AC - \beta. \quad (39)$$

Proof. Use (13),(14) to get (38),(39), respectively. ■

Proposition 4.2. *In the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$,*

$$\frac{[BA^2]}{q^2(q-q^{-1})^2} - \frac{(q+q^{-1})[CA]}{q(q-q^{-1})} = A^2B + (q+q^{-1})AC - A\gamma + \frac{r_5}{q(q-q^{-1})}, \quad (40)$$

$$\frac{[BAC]}{(q+q^{-1})(q-q^{-1})^2} = B^2 - A^2 + \frac{A\alpha - B\beta + r_1}{q+q^{-1}}. \quad (41)$$

Proof. Apply $-\text{ad } A$ to both sides of (38). The linear combination in the right side of the resulting equation contains ABA which can be further simplified using (13). From this, we get (40). We show (41) holds. Apply $-\text{ad } C$ to both sides of (38). The result involves ACB in the right side, which can be further simplified using (15). From this, we get (41). ■

Definition 4.3. *We define the following elements of \mathcal{L} .*

$$H_0 := \frac{[[CB], [BA]] - [BABC]}{(q+q^{-1})^2(q-q^{-1})^2} + \frac{[BA^2]}{q-q^{-1}} - 2[CA] \quad (42)$$

$$I_0 := [H_0, [BA]] \quad (43)$$

Lemma 4.4. *In the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$,*

$$\frac{[BA]\alpha}{(q+q^{-1})^2} = H_0 + \frac{r_1B - Ar_3}{(q+q^{-1})^2} + \frac{q(q-q^{-1})r_5}{q+q^{-1}}. \quad (44)$$

Proof. Apply $-\text{ad } B$ to both sides of (41). The left side involves BA^2 which can be written uniquely as a linear combination of $A^2B, AC, A\gamma, B, \beta$ by repeated use of the relations (13),(14). We get

$$\begin{aligned} \frac{[BACB]}{q^2(q+q^{-1})^2(q-q^{-1})^3} &= A^2B + q^{-3}(q^4+1)AC - A\gamma \\ &\quad - q^{-2}(q+q^{-1})(q-q^{-1})B + q^{-2}(q-q^{-1})\beta \\ &\quad + \frac{r_5}{q(q+q^{-1})} - \frac{[BA]\alpha + Ar_3 - r_1B}{q^2(q+q^{-1})^2(q-q^{-1})} \end{aligned} \quad (45)$$

In (45), eliminate $A^2B, A\gamma$ using (40), and eliminate AC, B, β using (39). In the resulting equation, express all Lie monomials in terms of Hall basis elements. From this, we get (44). ■

Lemma 4.5. *In the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$,*

$$I_0 = [BA] \frac{[r_0, B] - [r_3, A]}{(q+q^{-1})^2} - \frac{q(q-q^{-1})[r_5, [BA]]}{q+q^{-1}} + \frac{[Ar_3 - r_1B, [BA]]}{(q+q^{-1})^2}. \quad (46)$$

Proof. Apply $-\text{ad } [BA]$ to both sides of (44). The resulting left side is

$$\frac{[[BA]\alpha, [BA]]}{(q+q^{-1})^2}, \quad (47)$$

which can be simplified into

$$[BA] \frac{[r_0, B] - [r_3, A]}{(q + q^{-1})^2}$$

using $r_0 = [A, \alpha]$, $r_3 = [B, \alpha]$. The result is (46). ■

Theorem 4.6. *The Lie algebra $L = \mathcal{L}/(\mathbb{I} \cap \mathcal{L})$ is not freely generated by A, B, C .*

Proof. Observe that if we write I_0 in terms of Hall basis elements, we have

$$I_0 = \frac{[[[CB], [BA]], [BA]] - [[BABC], [BA]]}{(q + q^{-1})^2(q - q^{-1})^2} + \frac{[[BA^2], [BA]]}{q - q^{-1}} - 2[[CA], [BA]],$$

which, by the linear independence of the Hall basis elements, implies that $I_0 \neq 0$. But by (46), we have $I_0 \in \mathbb{I} \cap \mathcal{L}$. Therefore, $\mathcal{L}/(\mathbb{I} \cap \mathcal{L}) \neq \mathcal{L}$. ■

5 Properties of some standard Lie monomials of L

We discuss properties of some standard Lie monomials of L in relation to the filtration $\{\Delta_n\}_{n \in \mathbb{N}}$ of Δ .

Proposition 5.1. *For any $i, j \in \mathbb{N}$, the following hold in Δ .*

$$[BA^i] - q^i(q - q^{-1})^i A^i B \in \Delta_i, \quad (48)$$

$$[CA^i] - (-1)^i q^{-i}(q - q^{-1})^i A^i C \in \Delta_i, \quad (49)$$

$$[CB^j] - q^j(q - q^{-1})^j B^j C \in \Delta_j. \quad (50)$$

Proof. We show (48) holds by induction on i . The case $i = 0$ is trivial. Suppose that for some $i \in \mathbb{N}$, we have

$$[BA^{i-1}] - q^{i-1}(q - q^{-1})^{i-1} A^{i-1} B \in \Delta_{i-1}. \quad (51)$$

Denote the element in (51) by X . By the properties of the filtration $\{\Delta_n\}_{n \in \mathbb{N}}$, we have $[X, A] \in \Delta_i$. Using (48), we further obtain

$$[X, A] + q^i(q - q^{-1})^i A^{i-1}((q + q^{-1})C - \gamma) = [BA^i] - q^i(q - q^{-1})^i A^i B,$$

which proves (48). The relations (49) and (50) are proven similarly. ■

Proposition 5.2 ([7, Lemma 8.1]). *Let $i, j, k \in \mathbb{N}$. Then the following hold in Δ .*

$$[A, A^i B^j C^k] - (1 - q^{2(j-k)}) A^{i+1} B^j C^k \in \Delta_{i+j+k}, \quad (52)$$

$$[B, A^i B^j C^k] - (q^{2i} - q^{2k}) A^i B^{j+1} C^k \in \Delta_{i+j+k}, \quad (53)$$

$$[C, A^i B^j C^k] - (q^{2(j-i)} - 1) A^i B^j C^{k+1} \in \Delta_{i+j+k}. \quad (54)$$

Proposition 5.3. *For nonzero $i, j, k \in \mathbb{N}$, the following hold in Δ .*

$$[BA^i B^j] - (-1)^j q^i (q^{2i} - 1)^j (q - q^{-1})^i A^i B^{j+1} \in \Delta_{i+j}, \quad (55)$$

$$[CA^i C^k] - (-1)^i q^{-i(2k+1)} (q^{2i} - 1)^k (q - q^{-1})^i A^i C^{k+1} \in \Delta_{i+k}, \quad (56)$$

$$[CB^j C^k] - (-1)^k q^j (q^{2j} - 1)^k (q - q^{-1})^j B^j C^{k+1} \in \Delta_{j+k}. \quad (57)$$

Proof. To show (55), use the relation (48), the relation (53) with k set to zero, and induction on j . The relations (56) and (57) are proven similarly. ■

Proposition 5.4. *The complement Δ_1^c of Δ_1 in Δ_2 contains $[CAB]$ and $[BAC]$.*

Proof. Use the canonical map $\mathbb{F}\langle \mathcal{X} \rangle \rightarrow \Delta$ on (41) in order to obtain

$$\frac{[BAC]}{(q+q^{-1})(q-q^{-1})^2} = B^2 - A^2 + \frac{A\alpha - B\beta}{q+q^{-1}} \in \Delta_1^c. \quad (58)$$

Apply $-\rho^2$ to both sides of (58). We get

$$\frac{[CAB]}{(q+q^{-1})(q-q^{-1})^2} = C^2 - A^2 + \frac{A\alpha - C\gamma}{q+q^{-1}} \in \Delta_1^c. \quad \blacksquare$$

Proposition 5.5. *For nonzero $i, j, k \in \mathbb{N}$ with $i \geq 2$, the following hold in Δ .*

$$[BAB^j C^k] - (-1)^{j+k} q^{-j} (q^{2j} - 1)^k (q - q^{-1})^{j+1} B^j C^{k-1} \Omega \in \Delta_{j+k+1}, \quad (59)$$

$$[CA^i B^j] - (-1)^{i+j} q^{1-i} (q^{2(i-1)} - 1)^j (q - q^{-1})^i A^{i-1} B^{j-1} \Omega \in \Delta_{i+j}, \quad (60)$$

$$[CA^i B C^k] - (-1)^{i+1} q^{(1-i)(1+2k)} (q^{2(i-1)} - 1)^{k+1} (q - q^{-1})^i A^{i-1} C^k \Omega \in \Delta_{i+k+1}. \quad (61)$$

Proof. We show (59) holds. We first consider the case $k = 1$. By setting $i = 1$ in (55), we get

$$[BAB^j] - (-1)^j q^{j+1} (q - q^{-1})^{j+1} AB^{j+1} \in \Delta_{j+1}. \quad (62)$$

Apply $-\text{ad } C$ to the element in (62). We get

$$[BAB^j C] - (-1)^j q^{j+1} (q - q^{-1})^{j+1} [AB^{j+1}, C] \in \Delta_{j+2}. \quad (63)$$

In (54) set i, j, k to $1, j+1, 0$, respectively, and combine with (63). We have

$$[BAB^j C] - (-1)^{j+1} q^{j+1} (q^{2j} - 1) (q - q^{-1})^{j+1} AB^{j+1} C \in \Delta_{j+2}. \quad (64)$$

Using (13), it is routine to show that

$$AB^n - q^{-2n} B^n A \in \Delta_n, \quad (65)$$

for $n \in \mathbb{N}$. Set $n = j+1$ in (65) and multiply the element by C from the right. We get

$$AB^{j+1} C - q^{-2(j+1)} B^{j+1} AC \in \Delta_{j+2}. \quad (66)$$

From (64) and (66), we get

$$[BAB^j C] - (-1)^{j+1} q^{-(j+1)} (q^{2j} - 1) (q - q^{-1})^{j+1} B^{j+1} AC \in \Delta_{j+2}. \quad (67)$$

Using the fact that Ω is equal to (26), we have

$$B^{j+1} AC - q B^j \Omega \in \Delta_{j+2}. \quad (68)$$

From (67) and (68), we get

$$[BAB^j C] - (-1)^{j+1} q^{-j} (q^{2j} - 1) (q - q^{-1})^{j+1} B^j \Omega \in \Delta_{j+2}, \quad (69)$$

from which we see that (59) holds for $k = 1$ and for nonzero $j \in \mathbb{N}$. Using (54), (69) and induction on k , we find that (59) holds for nonzero $j, k \in \mathbb{N}$. We now show (60) holds. Since $i \geq 2$, we can rewrite (59) changing the exponents j, k to $i-1, j$, respectively.

$$[BAB^{i-1} C^j] - (-1)^{i+j-1} q^{1-i} (q^{2(i-1)} - 1)^j (q - q^{-1})^i B^{i-1} C^{j-1} \Omega \in \Delta_{i+j} \quad (70)$$

Denote the element in (70) by X . Since Δ_{i+j} is invariant under ρ , we have $-\rho^2(X) \in \Delta_{i+j}$, where $-\rho^2(X)$ is the element in (60). Thus, (60) holds for nonzero $i, j \in \mathbb{N}$. Finally, we show (61) holds. Set $j = 1$ in (60).

$$[CA^i B] - (-1)^{i+1} q^{1-i} (q^{2(i-1)} - 1) (q - q^{-1})^i A^{i-1} \Omega \in \Delta_{i+1} \quad (71)$$

Since Ω is central, if we apply $-\text{ad } C$ to the element in (71), we get

$$[CA^i BC] - (-1)^{i+1} q^{1-i} (q^{2(i-1)} - 1) (q - q^{-1})^i [A^{i-1}, C] \Omega \in \Delta_{i+2}. \quad (72)$$

From (54) we obtain

$$[A^{i-1}, C] \Omega - q^{-2(i-1)} (q^{2(i-1)} - 1) A^{i-1} C \Omega \in \Delta_{i+2}. \quad (73)$$

From (71) and (73),

$$[CA^i BC] - (-1)^{i+1} q^{(1-i) \cdot 3} (q^{2(i-1)} - 1)^2 (q - q^{-1})^i A^{i-1} C \Omega \in \Delta_{i+2}, \quad (74)$$

from which we see that (61) holds for $k = 1$. Using (54), (74) and induction on k , we find that (61) holds for nonzero $i, k \in \mathbb{N}$ with $i \geq 2$. ■

Proposition 5.6. *The following hold in Δ .*

$$[CA^2 B] + (q - q^{-1})^3 A \Omega \in \Delta_3, \quad (75)$$

$$[BABC] - (q - q^{-1})^3 B \Omega \in \Delta_3, \quad (76)$$

$$[[CB], [CA]] - (q - q^{-1})^3 C \Omega \in \Delta_3. \quad (77)$$

Proof. The relations (75), (76) follow from (60), (59), respectively. We show (77) holds. Let $V := -[BAC^2] + [CABC]$. By Proposition 5.4, we have $V \in \Delta_3$. Denote the element in (76) by X . Using the fact that Δ_3 is invariant under ρ , we have

$$\rho(X) = [CBCA] - (q - q^{-1})^3 C \Omega \in \Delta_3.$$

Using the Jacobi identity to express $[CBCA]$ in terms of standard Lie monomials, we further have

$$\rho(X) = [[CB], [CA]] - V - (q - q^{-1})^3 C \Omega \in \Delta_3,$$

and it follows that

$$[[CB], [CA]] - (q - q^{-1})^3 C \Omega = V + \rho(X) \in \Delta_3. \quad \blacksquare$$

6 The standard Lie monomials of L of length at most 4

Recall that the span of the standard Lie monomials of L of length at most n is L_n . Our goal in this section is to show that the standard Lie monomials of L of length at most 4 are linearly independent, and hence form a basis for L_4 .

Proposition 6.1. *For nonzero $j, k \in \mathbb{N}$, the following hold in Δ .*

$$[BAB^j] - (-1)^j q^{(j+1)} (q - q^{-1})^{j+1} AB^{j+1} \in \Delta_{j+1}, \quad (78)$$

$$[CAC^k] + q^{-(k+2)} (q - q^{-1})^{k+1} AC^{k+1} \in \Delta_{k+1}, \quad (79)$$

$$[CBC^k] - (-1)^k q^{(k+1)} (q - q^{-1})^{k+1} BC^{k+1} \in \Delta_{k+1}, \quad (80)$$

$$[BA^2 B^j] - (-1)^j q^{2(j+1)} (q + q^{-1})^j (q - q^{-1})^{j+2} A^2 B^{j+1} \in \Delta_{j+2}, \quad (81)$$

$$[CA^2 C^k] - q^{-2(k+1)} (q + q^{-1})^k (q - q^{-1})^{k+1} A^2 C^{k+1} \in \Delta_{k+2}, \quad (82)$$

$$[CB^2 C^k] - (-1)^k q^{2(k+1)} (q + q^{-1})^k (q - q^{-1})^{k+2} B^2 C^{k+1} \in \Delta_{k+2}. \quad (83)$$

Proof. Set $i = 1, 2$ in (55) to get (78), (81). Do similarly to (56) and (57) in order to show the other relations. ■

Lemma 6.2. *Fix a nonzero $n \in \mathbb{N}$. The following vectors are linearly independent in Δ for any $i, j, k \in \mathbb{N}$ such that $1 \leq i, j, k \leq n$.*

$$1, A, B, C, \quad (84)$$

$$[CAB], [BAC], \quad (85)$$

$$[CA^2 B], [BABC], [[CB], [CA]], \quad (86)$$

$$[BA^i], [BAB^j], [BA^2 B^j], \quad (87)$$

$$[CA^i], [CAC^k], [CA^2 C^k], \quad (88)$$

$$[CB^j], [CBC^k], [CB^2 C^k]. \quad (89)$$

Proof. Fix $n \in \mathbb{N}$. It suffices to show that there exists an upper triangular transition matrix from the above vectors to a subset of the basis of Δ consisting of the vectors in (27):

$$A^i B^j C^k \Omega^l \alpha^r \beta^s \gamma^t, \quad i, j, k, l, r, s, t \in \mathbb{N}, \quad ijk = 0.$$

Let $i, j, k \in \mathbb{N}$ such that $1 \leq i, j, k \leq n$. From Propositions 5.1, 5.4, 5.6, 6.1 we have the following data:

$$[CAB] - c_1 C^2 - d_1 A^2 - d_2 C\gamma - d_3 A\alpha \in \Delta_0, \quad (90)$$

$$[BAC] - c_2 B^2 - d_4 A^2 - d_5 B\beta - d_6 A\alpha \in \Delta_0, \quad (91)$$

$$[CA^2 B] - c_3 A\Omega \in \Delta_3, \quad (92)$$

$$[BABC] - c_4 B\Omega \in \Delta_3, \quad (93)$$

$$[[CB], [CA]] - c_5 C\Omega \in \Delta_3, \quad (94)$$

$$[BA^i] - e_i A^i B \in \Delta_i, \quad (95)$$

$$[CA^i] - f_i A^i C \in \Delta_i, \quad (96)$$

$$[CB^j] - g_j B^j C \in \Delta_j, \quad (97)$$

$$[BAB^j] - e'_j AB^{j+1} \in \Delta_{j+1}, \quad (98)$$

$$[CAC^k] - f'_k AC^{k+1} \in \Delta_{k+1}, \quad (99)$$

$$[CBC^k] - g'_k BC^{k+1} \in \Delta_{k+1}, \quad (100)$$

$$[BA^2 B^j] - e''_j A^2 B^{j+1} \in \Delta_{j+2}, \quad (101)$$

$$[CA^2 C^k] - f''_k A^2 C^{k+1} \in \Delta_{k+2}, \quad (102)$$

$$[CB^2 C^k] - g''_k B^2 C^{k+1} \in \Delta_{k+2}, \quad (103)$$

where the small letters (other than i, j, k) denote scalars. Each of (92) to (103) is of the form $M - aV \in \Delta_m$, where M is a Lie monomial, $a \in \mathbb{F}$, and V is an element of the basis of Δ consisting of the vectors in (27), and $V \notin \Delta_m$. Call V the *leading term* of M . For (90), (91), define the leading terms of $[CAB]$, $[BAC]$ by C^2, B^2 , respectively. Observe that no two distinct Lie monomials found in (90) to (103) have the same leading terms. This yields a transition matrix from the vectors (84) to (89) to some of the vectors in (27) such that all entries below the main diagonal are zero, and that the diagonal entries are

$$c_1, \dots, c_5, e_i, f_i, g_j, e'_j, f'_k, g'_k, e''_j, f''_k, g''_k.$$

By Propositions 5.1, 5.4, 5.6 and 6.1, all such scalars are nonzero. Hence, the transition matrix is upper triangular. \blacksquare

Notation 6.3. Let \mathcal{I}_n denote the set consisting of all the linearly independent vectors in Lemma 6.2.

Lemma 6.4. Fix nonzero $m, n \in \mathbb{N}$. The vectors $X\alpha^r \beta^s \gamma^t$ are linearly independent in Δ for any $X \in \mathcal{I}_n$ and any $r, s, t \in \mathbb{N}$ such that $r + s + t \leq m$.

Proof. The proof is similar to that of Lemma 6.2, but with (90) to (103) modified as follows. For each of (90) to (103), multiply the element by $\alpha^r \beta^s \gamma^t$ and add $r + s + t$ to the index of the filtration subspace. Based on these new data, an upper triangular transition matrix can be constructed. \blacksquare

Notation 6.5. Let \mathcal{I}_n^m denote the set consisting of all the linearly independent vectors in Lemma 6.4. Observe that the vectors

$$[CB]\gamma, [BA]\beta, [CA]\alpha, [CB]\beta, [CA]\gamma, [BA]\alpha, \quad (104)$$

are in \mathcal{I}_3^1 . Let \mathcal{I}^* denote the set obtained from \mathcal{I}_3^1 by replacing the vectors in (104) by the vectors

$$[CAB^2], [BA^2 C], [CABC], [BAC^2], [[CA], [BA]], [[CB], [BA]]. \quad (105)$$

Proposition 6.6. *The following hold in Δ .*

$$\frac{[BA]\alpha}{(q+q^{-1})^2} = \frac{[[CB], [BA]] - [BABC]}{(q+q^{-1})^2(q-q^{-1})^2} + \frac{[BA^2]}{q-q^{-1}} - 2[CA], \quad (106)$$

$$\frac{[CB]\beta}{(q+q^{-1})^2} = \frac{[BAC^2] - [CABC]}{(q+q^{-1})^2(q-q^{-1})^2} + \frac{[CB^2]}{q-q^{-1}} + 2[BA], \quad (107)$$

$$\frac{[CA]\gamma}{(q+q^{-1})^2} = -\frac{[[CA], [BA]] + [CA^2B]}{(q+q^{-1})^2(q-q^{-1})^2} + \frac{[CAC]}{q-q^{-1}} - 2[CB], \quad (108)$$

$$\frac{[BA]\beta}{(q+q^{-1})^2} = \frac{[[CA], [BA]] - [BA^2C]}{(q+q^{-1})^2(q-q^{-1})^2} - \frac{[BAB]}{q-q^{-1}} + 2[CB], \quad (109)$$

$$\frac{[CB]\gamma}{(q+q^{-1})^2} = \frac{-[[CB], [BA]] + [BABC] - [CAB^2]}{(q+q^{-1})^2(q-q^{-1})^2} - \frac{[CBC]}{q-q^{-1}} - 2[CA], \quad (110)$$

$$\frac{[CA]\alpha}{(q+q^{-1})^2} = \frac{-[CABC]}{(q+q^{-1})^2(q-q^{-1})^2} - \frac{[CA^2]}{q-q^{-1}} - 2[BA]. \quad (111)$$

Proof. Apply the canonical map $\mathbb{F}\langle \mathcal{X} \rangle \rightarrow \Delta$ to both sides of (44) to get (106). Apply ρ, ρ^2 to both sides of (106) to get (107), (108), respectively. To get (109), apply σ to both sides of (106). Apply ρ, ρ^2 to both sides of (109) to get (110), (111), respectively. ■

Lemma 6.7. *The vectors in \mathcal{I}^* are linearly independent in Δ .*

Proof. We use (106) to (111) to construct a transition matrix from the elements of \mathcal{I}_3^1 to the elements of \mathcal{I}^* . Denote such transition matrix by T . Order the rows of T such that the last 17 correspond to

$$[BA], [CA], [CB], [BA^2], [CA^2], [BAB], \quad (112)$$

$$[CB^2], [CAC], [CBC], [CA^2B], [BABC], \quad (113)$$

$$[CAB^2], [BA^2C], [CABC], [BAC^2], [[CA], [BA]], [[CB], [BA]], \quad (114)$$

in that order, while order the columns of T such that the last 17 correspond to the vectors in (112), (113) together with

$$[CB]\gamma, [BA]\beta, [CA]\alpha, [CB]\beta, [CA]\gamma, [BA]\alpha. \quad (115)$$

Observe that all the vectors in \mathcal{I}_3^1 to be replaced to form \mathcal{I}^* are in (115), all the replacements are in (114), and all the other vectors that appear in (106) to (111) (which we use to construct the transition matrix) appear in (112), (113). Then T is of the form

$$T = \left[\begin{array}{c|c} I & M \\ \hline 0 & T' \end{array} \right]$$

where I is an identity matrix, M is some matrix with 7 columns, and T' is a 7×7 matrix which has the following properties. All diagonal entries of T' are nonzero. Denote the ij -entry of T' by T'_{ij} . All entries of T' below the main diagonal and all entries in the first two rows are zero except the ones that appear below:

$$\frac{1}{(q-q^{-1})^2} = -T'_{11} = T'_{31} = -T'_{71} = -T'_{22} = T'_{62} \neq 0.$$

By these observations about T' , we find that T is invertible. This implies that the vectors in \mathcal{I}^* are linearly independent. ■

Theorem 6.8. *The standard Lie monomials of L of length at most 4 form a basis for L_4 .*

Proof. All such vectors are in \mathcal{I}^* . Use Lemma 6.7 and the fact that the vectors in the statement span L_4 . ■

7 The standard Lie monomials of L of length at most 5

In this section, we show the Lie algebra relations that hold in L_5 . We also exhibit a basis for L if q is not a sixth root of unity.

Lemma 7.1. *The following hold in Δ .*

$$\begin{aligned} [CA] \beta + [BA] \gamma &= \frac{-[[BA^2], [CA]] + [[CA^2], [BA]]}{(q - q^{-1})^3} - \frac{[CA^2C] + [BA^2B]}{(q - q^{-1})^2} \\ &\quad - \frac{(q + q^{-1})^2 ([BAC] - [CAB])}{q - q^{-1}} \end{aligned} \quad (116)$$

$$\begin{aligned} [CB] \alpha - [BA] \gamma &= \frac{-[[BAB], [CB]] + [[CB^2], [BA]]}{(q - q^{-1})^3} - \frac{[CB^2C] - [BA^2B]}{(q - q^{-1})^2} \\ &\quad - \frac{(q + q^{-1})^2 [CAB]}{q - q^{-1}} \end{aligned} \quad (117)$$

Proof. In view of Remark 3.3, write each of the left and right sides of (116) as a linear combination of irreducible Δ -words. This yields the same linear combination of the basis vectors (16) of Δ . Apply ρ to both sides of (116) to get (117). ■

Theorem 7.2. *The following relations hold in L .*

$$\begin{aligned} \frac{[BA^2BC]}{q - q^{-1}} &= -\frac{-(2q^2 + 1)(q^2 + 2) ([[BA^2], [CB]] + [[BAB], [CA]])}{2q^2(q + q^{-1})^2(q - q^{-1})} \\ &\quad - \frac{(q^4 + 3q^2 + 1) ([BAC], [BA]] - 2[[CAB], [BA]])}{2q^2(q + q^{-1})^2(q - q^{-1})} \\ &\quad - [BAC^2] + 2[CABC] - [BAB^2] + [BA^3], \end{aligned} \quad (118)$$

$$\begin{aligned} \frac{[[CB^2], [CA]]}{2(q + q^{-1})^2(q - q^{-1})} &= \frac{-(3q^4 + 5q^2 + 3) [[BAC], [CB]]}{2q^2(q + q^{-1})^2(q - q^{-1})} \\ &\quad + \frac{(q^4 + 3q^2 + 1) [[CAB], [CB]]}{2q^2(q + q^{-1})^2(q - q^{-1})} \\ &\quad - \frac{(2q^4 + 3q^2 + 2) [[CBC], [BA]]}{2q^2(q + q^{-1})^2(q - q^{-1})} \\ &\quad - \frac{[BABC^2] - [CAB^2C]}{q - q^{-1}} \\ &\quad + [CBC^2] + [BA^2C] - [CB^3] + [CA^2B], \end{aligned} \quad (119)$$

$$\begin{aligned} \frac{[[CA^2], [CB]]}{2(q + q^{-1})^2(q - q^{-1})} &= \frac{-2 [[BAC], [CA]] + [[CAB], [CA]]}{2q^2(q + q^{-1})^2(q - q^{-1})} \\ &\quad + \frac{(2q^2 + 1)(q^2 + 2) [[CAC], [BA]]}{2q^2(q + q^{-1})^2(q - q^{-1})} \\ &\quad + \frac{[CA^2BC]}{q - q^{-1}} - 2 [[CB], [BA]] - [CAC^2] \\ &\quad + 2[BABC] - [CAB^2] + [CA^3], \end{aligned} \quad (120)$$

$$\begin{aligned} [[CAC], [CB]] &= -[[BAB], [CB]] + [[CBC], [CA]] - [[BA^2], [CA]] \\ &\quad + [[CB^2], [BA]] + [[CA^2], [BA]]. \end{aligned} \quad (121)$$

Proof. To show (118), we first show that the equation

$$\begin{aligned} \frac{[BA^2BC]}{(q+q^{-1})^2(q-q^{-1})^2} &= \frac{-[BAC^2] + 2[CABC]}{q-q^{-1}} \\ &+ \frac{(q^4+1)([BAB^2] - [BA^3])}{q^2(q-q^{-1})} \\ &- \frac{2(q^6-1)([CB^2] - [CA^2])}{q^3(q-q^{-1})} \\ &+ \frac{[BAC]\gamma}{(q+q^{-1})^2} + [BAB]\beta + [BA^2]\alpha \end{aligned} \quad (122)$$

holds in Δ . In view of Remark 3.3, write each of the left and right sides of (122) as a linear combination of irreducible Δ -words. This yields the same linear combination of the basis vectors (16) of Δ . Apply $-\text{ad } A, -\text{ad } B, -\text{ad } B, -\text{ad } C$ to both sides of (106),(108),(109),(110), respectively. Write all Lie monomials in standard form. We get

$$[BA^2]\alpha = f_1, \quad (123)$$

$$[CA^2]\beta = f_2, \quad (124)$$

$$[CAB]\gamma = f_3, \quad (125)$$

$$[BAC]\gamma + k[CAB]\gamma = f_4, \quad (126)$$

for some $k \in \mathbb{F}$ and some $f_1, f_2, f_3, f_4 \in L$. Eliminate $[BA^2]\alpha, [CA^2]\beta, [BAC]\gamma$ in (122) using (123) to (126). The result is (118). Apply ρ, ρ^2 to both sides of (118) in order to obtain (119),(120), respectively. We now show (121) holds. Add (116) and (117). We get

$$[CB]\alpha + [CA]\beta = g_1 + g_2, \quad (127)$$

where g_1, g_2 are the right sides of (116),(117), respectively. Apply $-\rho$ to both sides of (117). We get

$$[CB]\alpha + [CA]\beta = g_3, \quad (128)$$

for some $g_3 \in L$ such that $[[CAC], [CB]]$ appears with nonzero coefficient in g_3 . Eliminate $[CB]\alpha + [CA]\beta$ in (127),(128). We get (121) as desired. \blacksquare

At this point we have shown that each of

$$[BA^2BC], [[CB^2], [CA]], [[CA^2], [CB]], [[CAC], [CB]], \quad (129)$$

is linearly dependent on standard Lie monomials of length at most 5 that are not in (129). In what follows, we shall show that the standard Lie monomials of length at most 5 except (129) are linearly independent in Δ .

Proposition 7.3. *The following hold in Δ .*

$$[CA^3B] - (q+q^{-1})(q-q^{-1})^4 A^2\Omega \in \Delta_4, \quad (130)$$

$$[CA^2B^2] - q(q-q^{-1})^4 AB\Omega \in \Delta_4, \quad (131)$$

$$[CA^2BC] + q^{-1}(q-q^{-1})^4 AC\Omega \in \Delta_4, \quad (132)$$

$$[BAB^2C] + (q+q^{-1})(q-q^{-1})^4 B^2\Omega \in \Delta_4, \quad (133)$$

$$[BABC^2] + q(q-q^{-1})^4 BC\Omega \in \Delta_4, \quad (134)$$

$$[[CBC], [CA]] + (q+q^{-1})(q-q^{-1})^4 C^2\Omega \in \Delta_4. \quad (135)$$

Proof. Use Proposition 5.5 to show (130) to (134) hold. To show (135), set $i = 2$ in (49). We get

$$[CA^2] - q^{-2}(q-q^{-1})^2 A^2C \in \Delta_2. \quad (136)$$

Applying $-\rho^2$ to the element in (136) and using the fact that Δ_2 is invariant under ρ , we have

$$[CBC] + q^{-2}(q - q^{-1})^2 C^2 B \in \Delta_2. \quad (137)$$

Set $i = 1$ in (49). We get

$$[CA] + q^{-1}(q - q^{-1})AC \in \Delta_1. \quad (138)$$

By taking the Lie bracket of the elements in (137) and (138), we have

$$[[CBC], [CA]] - q^{-3}(q - q^{-1})^3 (C^2 BAC - AC^3 B) \in \Delta_3 \subset \Delta_4. \quad (139)$$

Using (14), it is routine to show that

$$C^2 A - q^{-4} AC^2 \in \Delta_2,$$

from which we obtain

$$(q - q^{-1})^3 (qC^2 ACB - q^{-3} AC^3 B) \in \Delta_4. \quad (140)$$

Combining (139) and (140), we obtain

$$[[CBC], [CA]] - (q - q^{-1})^3 C^2 (q^{-3} BAC - qACB) \in \Delta_4. \quad (141)$$

Using the fact that (25) and (26) are both equal to Ω , we have

$$(q + q^{-1})(q - q^{-1})\Omega + q^{-3} BAC - qACB \in \Delta_2. \quad (142)$$

Finally, we get (135) from (141) and (142). \blacksquare

Proposition 7.4. *For nonzero $j, k \in \mathbb{N}$, the following hold in Δ .*

$$[BA^3 B^j] - (-1)^j q^3 (q^6 - 1)^j (q - q^{-1})^3 A^3 B^{j+1} \in \Delta_{j+3}, \quad (143)$$

$$[CA^3 C^k] + q^{-3(2k+1)} (q^6 - 1)^k (q - q^{-1})^3 A^3 C^{k+1} \in \Delta_{k+3}, \quad (144)$$

$$[CB^3 C^k] - (-1)^k q^3 (q^6 - 1)^k (q - q^{-1})^3 B^3 C^{k+1} \in \Delta_{k+3}. \quad (145)$$

Proof. Use Proposition 5.3. \blacksquare

Lemma 7.5. *Assume q is not a sixth root of unity. Fix a nonzero $n \in \mathbb{N}$. The following vectors are linearly independent in Δ for any $i, j, k \in \mathbb{N}$ such that $1 \leq i, j, k \leq n$.*

$$1, A, B, C, \quad (146)$$

$$[CAB], [BAC], \quad (147)$$

$$[CA^2 B], [BABC], [[CB], [CA]], \quad (148)$$

$$[CA^3 B], [CA^2 B^2], [CA^2 BC], \quad (149)$$

$$[BAB^2 C], [BABC^2], [[CBC], [CA]], \quad (150)$$

$$[BA^i], [BAB^j], [BA^2 B^j], [BA^3 B^j], \quad (151)$$

$$[CA^i], [CAC^k], [CA^2 C^k], [CA^3 C^k], \quad (152)$$

$$[CB^j], [CBC^k], [CB^2 C^k], [CB^3 C^k]. \quad (153)$$

Proof. The proof is similar to that of Lemma 6.2. In order to construct the desired upper triangular transition matrix, we combine the data from (90) to (103) to that in (130) to (135), and (143) to (145). Recall that the transition matrix that can be constructed from the data is upper triangular if the scalar coefficients of the leading terms are nonzero. Those in (90) to (103) are nonzero as shown in the proof of Lemma 6.2. The scalar coefficients in (130) to (135) are nonzero by the manner q is defined. Finally, the scalar coefficients in (143) to (145) are nonzero since q is further assumed to be not a sixth root of unity. \blacksquare

Notation 7.6. *Let \mathcal{J}_n denote the set consisting of all the linearly independent vectors in Lemma 7.5.*

Lemma 7.7. Assume q is not a sixth root of unity. Fix nonzero $m, n \in \mathbb{N}$. The vectors $Y\alpha^r\beta^s\gamma^t$ are linearly independent in Δ for any $Y \in \mathcal{J}_n$ and any $r, s, t \in \mathbb{N}$ such that $r + s + t \leq m$.

Proof. The proof is similar to that of Lemma 6.2. For each of the data used in the proof of Lemma 7.5, which are (90) to (103), (130) to (135), and (143) to (145), multiply the element by $\alpha^r\beta^s\gamma^t$ and add $r + s + t$ to the index of the filtration subspace. Use these new data to construct a similar upper triangular transition matrix. \blacksquare

Notation 7.8. Let \mathcal{J}_n^m denote the set consisting of all the linearly independent vectors in Lemma 7.7. Observe that the vectors

$$[CB]\gamma, [BA]\beta, [CA]\alpha, [CB]\beta, [CA]\gamma, [BA]\alpha, \quad (154)$$

are in \mathcal{J}_4^1 . Let \mathcal{J}^* denote the set obtained from \mathcal{J}_4^1 by replacing the vectors in (154) by the following vectors

$$[CAB^2], [BA^2C], [CABC], [BAC^2], [[CA], [BA]], [[CB], [BA]]. \quad (155)$$

Lemma 7.9. Assume q is not a sixth root of unity. The vectors in \mathcal{J}^* are linearly independent in Δ .

Proof. The proof is similar to that of Lemma 6.7. \blacksquare

Proposition 7.10. Assume q is not a sixth root of unity. If V is a subspace of $\text{Span } \mathcal{J}^*$ such that

$$\text{Span } \mathcal{J}^* = V + \text{Span } \mathcal{J}_4^0, \quad (\text{direct sum})$$

then a basis for V is $\mathcal{J}^* \setminus \mathcal{J}_4^0$.

Proof. This follows from the fact that $\mathcal{J}^*, \mathcal{J}_4^0$ are both linearly independent sets and that $\mathcal{J}_4^0 \subset \mathcal{J}^*$. \blacksquare

Lemma 7.11. The following hold in Δ .

$$[CAC]\alpha + \frac{[CABC^2]}{(q - q^{-1})^2} \in \text{Span } \mathcal{J}_4^0, \quad (156)$$

$$[CBC]\beta - \frac{[BAC^3]}{(q - q^{-1})^2} + \frac{[CABC^2]}{(q - q^{-1})^2} \in \text{Span } \mathcal{J}_4^0, \quad (157)$$

$$[BA^2]\gamma - \frac{(q^6 - 1)[[BA^2], [BA]]}{q^3(q - q^{-1})^3} \in \text{Span } \mathcal{J}_4^0, \quad (158)$$

$$[BAB]\gamma + \frac{(q + q^{-1})^2[[BAB], [BA]]}{(q - q^{-1})^2} \in \text{Span } \mathcal{J}_4^0, \quad (159)$$

$$[CA^2]\beta - \frac{(q^6 - 1)[[CA^2], [CA]]}{q^3(q - q^{-1})^3} \in \text{Span } \mathcal{J}_4^0, \quad (160)$$

$$[CAC]\beta + \frac{(q + q^{-1})^2[[CAC], [CA]]}{(q - q^{-1})^2} \in \text{Span } \mathcal{J}_4^0, \quad (161)$$

$$[CB^2]\alpha - \frac{(q^6 - 1)[[CB^2], [CB]]}{q^3(q - q^{-1})^3} \in \text{Span } \mathcal{J}_4^0, \quad (162)$$

$$[CBC]\alpha + \frac{(q + q^{-1})^2[[CBC], [CB]]}{(q - q^{-1})^2} \in \text{Span } \mathcal{J}_4^0. \quad (163)$$

Proof. Apply $-\text{ad } C$ to both sides of (111), (107) and write all Lie monomials in standard form in order to get (156), (157), respectively. To get (158), (159), we first show that the equation

$$\begin{aligned} \frac{[BA^2]\gamma}{2(q + q^{-1})^2} &= \frac{(q^6 - 1)[[BA^2], [BA]]}{2q^3(q + q^{-1})^2(q - q^{-1})^3} - \frac{[BA^3B]}{2(q + q^{-1})^2(q - q^{-1})^2} \\ &\quad - \frac{[[CA], [BA]] + [BA^2C]}{2(q - q^{-1})} + [CAC] - [BAB] \end{aligned} \quad (164)$$

holds in Δ . In view of Remark 3.3, write each of the left and right sides of (164) as a linear combination of irreducible Δ -words. This yields the same linear combination of the basis vectors (16) of Δ . This proves (164), from which (158) follows. To prove (159), apply σ to both sides of (164). We obtain

$$\begin{aligned} \frac{[BAB]\gamma}{2(q+q^{-1})^2} &= \frac{-[[BAB], [BA]]}{2(q-q^{-1})^2} - \frac{[BA^2B^2]}{2(q+q^{-1})^2(q-q^{-1})^2} \\ &\quad + \frac{[[CB], [BA]] + [BABC]}{2(q-q^{-1})} - [CBC] - [BA^2], \end{aligned} \quad (165)$$

from which (159) follows. Finally, to prove (160) to (163), apply ρ, ρ^2 to both sides of (164), (165). \blacksquare

Notation 7.12. Observe that the vectors

$$[CAC]\alpha, [CBC]\beta, [BA^2]\gamma, [BAB]\gamma, \quad (166)$$

$$[CA^2]\beta, [CAC]\beta, [CB^2]\alpha, [CBC]\alpha, \quad (167)$$

are in \mathcal{J}^* . Let \mathcal{K}_0 denote the set obtained from \mathcal{J}^* by replacing the vectors in (166), (167) by the vectors

$$[CABC^2], [BAC^3], [[BA^2], [BA]], [[BAB], [BA]], \quad (168)$$

$$[[CA^2], [CA]], [[CAC], [CA]], [[CB^2], [CB]], [[CBC], [CB]]. \quad (169)$$

Lemma 7.13. Assume q is not a sixth root of unity. The vectors in \mathcal{K}_0 are linearly independent in Δ .

Proof. We use Lemma 7.11 in order to obtain a transition matrix from the vectors in \mathcal{J}^* into the vectors in \mathcal{K}_0 . Denote each of (156) to (163) by

$$M_i + f_i \in \text{Span } \mathcal{J}_4^0,$$

where $1 \leq i \leq 8$, the vector M_i is an element of \mathcal{J}^* that is to be replaced in order to form \mathcal{K}_0 , while the standard Lie monomials that appear in f_i are the replacements. Using the usual ordering of standard Lie monomials, define \overline{M}_i as the largest standard Lie monomial in f_i . Let $j, k \in \mathbb{N}$, with $1 \leq j, k \leq 8$. Observe that if $M_j \neq M_k$ then $\overline{M}_j \neq \overline{M}_k$. Observe also that $M_i \in \mathcal{J}^* \setminus \mathcal{J}_4^0$ for $1 \leq i \leq 8$. By Proposition 7.10, the coefficient of M_i in f_i is -1 for all i . By these observations, it follows that there exists an upper triangular transition matrix from the vectors in \mathcal{J}^* into the vectors in \mathcal{K}_0 with nonzero diagonal entries. These diagonal entries are precisely the scalar coefficients of \overline{M}_i for all i . These coefficients are all nonzero since q is assumed to be not a sixth root of unity. Since the vectors in \mathcal{J}^* are linearly independent, the existence of a transition matrix just described implies that the vectors in \mathcal{K}_0 are also linearly independent. \blacksquare

Theorem 7.14. Assume q is not a sixth root of unity. The standard Lie monomials of L of length at most 5 except the vectors from (129) form a basis for L_5 .

Proof. Let \mathcal{K} denote the set obtained from \mathcal{K}_0 by replacing the vectors

$$[BAC]\gamma, [CAB]\gamma, [BAB]\beta, [BA^2]\alpha, \quad (170)$$

$$[BA^2]\beta, [CA^2]\gamma, [CA]\beta, \quad (171)$$

$$[BAC]\beta, [CAC]\gamma, [CA^2]\alpha, [CAB]\beta, \quad (172)$$

$$[CB^2]\gamma, [CB]\alpha, [BAB]\alpha, \quad (173)$$

$$[CAB]\alpha, [CB^2]\beta, [BAC]\alpha, [CBC]\gamma, \quad (174)$$

by the vectors

$$[[CAB], [BA]], [[BAC], [BA]], [[BAB], [CA]], [[BA^2], [CB]], \quad (175)$$

$$[BA^3C], [[CA^2], [BA]], [[BA^2], [CA]], \quad (176)$$

$$[BA^2C^2], [[CAC], [BA]], [[CAB], [CA]], [[BAC], [CA]], \quad (177)$$

$$[CAB^3], [[CB^2], [BA]], [[BAB], [CB]], \quad (178)$$

$$[CAB^2C], [[CBC], [BA]], [[CAB], [CB]], [[BAC], [CB]]. \quad (179)$$

We claim that \mathcal{K} is linearly independent. Observe that all vectors mentioned in the statement of the theorem are in \mathcal{K} . By Theorem 7.2, the vectors in (129) are linearly dependent on these vectors. The result follows. We now prove our claim. We construct five more sets in a manner similar to the construction of \mathcal{J}^* from \mathcal{J}_4^1 and to that of \mathcal{K}_0 from \mathcal{J}^* . The goal is that at each step, we prove that the constructed set is linearly independent. Let \mathcal{K}_1 denote the set obtained from \mathcal{K}_0 by replacing the vectors (170) in \mathcal{K}_0 by the vectors (175). Let \mathcal{K}_2 denote the set obtained from \mathcal{K}_1 in a similar manner until \mathcal{K}_5 , which is obtained from \mathcal{K}_4 by replacing the vectors (174) in \mathcal{K}_4 by the vectors (179). Observe that $\mathcal{K}_5 = \mathcal{K}$. We show that each of $\mathcal{K}_1, \dots, \mathcal{K}_5$ is a linearly independent set in Δ . Apply $-\text{ad } A, -\text{ad } B, -\text{ad } B, -\text{ad } A$ to both sides of (110),(108),(109),(111), respectively. Write all Lie monomials in standard form. We get

$$[BAC]\gamma + k[CAB]\gamma = f_1, \quad (180)$$

$$[CAB]\gamma = f_2, \quad (181)$$

$$[BAB]\beta = f_3, \quad (182)$$

$$[BA^2]\alpha = f_4, \quad (183)$$

for some $k \in \mathbb{F}$ and some $f_1, f_2, f_3, f_4 \in L$. Eliminate $[BA^2BC]$ in (180),(182),(183) using the relation (118). Solve the resulting system in order to obtain

$$[BAC]\gamma = g_1, \quad (184)$$

$$[CAB]\gamma = g_2, \quad (185)$$

$$[BAB]\beta = g_3, \quad (186)$$

$$[BA^2]\alpha = g_4, \quad (187)$$

where each of $g_1, \dots, g_4 \in L$, is a linear combination of

$$[[CAB], [BA]], [[BAC], [BA]], [[BAB], [CA]], [[BA^2], [CB]],$$

together with the vectors in \mathcal{K}_0 . We use the equations (184) to (187) to construct a transtion matrix T_1 from the vectors in \mathcal{K}_0 into those in \mathcal{K}_1 . Index the columns of T_1 such that the last 4 correspond to (170), while index the rows such that the last 4 correspond to (175). We find that T_1 has the form

$$T_1 = \left[\begin{array}{c|c} I_1 & U_1 \\ \hline 0 & L_1 \end{array} \right]$$

where I_1 is an identity matrix, U_1 is some matrix with 4 columns, and L_1 is a 4×4 matrix, with

$$\det L_1 = -\frac{1}{2(q+q^{-1})^2(q-q^{-1})^8} \neq 0,$$

which implies that $\det T_1 \neq 0$. Thus, \mathcal{K}_1 is linearly independent. For $2 \leq i \leq 5$, we can also construct a transtion matrix T_i from the vectors in \mathcal{K}_{i-1} into those in \mathcal{K}_i in a similar manner. The equations that can be used to construct T_i are also derived from (106) to (111) with the application of the appropriate map $\text{ad } X$ where $X \in \{A, B, C\}$. Furthermore, we find that T_i can be partitioned into four matrices similar to T_1 , and that if we denote the bottom right partition as L_i , then

$$\begin{aligned} \frac{1}{(q-q^{-1})^7} &= -\det L_2 = \det L_4 \neq 0, \\ \frac{1}{(q-q^{-1})^8} &= -\det L_3 = -\det L_5 \neq 0, \end{aligned}$$

which imply that T_i is invertible for $2 \leq i \leq 5$. Therefore, $\mathcal{K}_5 = \mathcal{K}$ is a linearly independent set in Δ . \blacksquare

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